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Self-similarity and probability distributions of turbulent intermittency

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Turbulent intermittency is analyzed in terms of statistics of breakdown coefficients, an alternative to the multifractal formalism. The processing of experimental data at a very large Reynolds number shows a persistent influence of the local Reynolds number up to a sufficiently large value. The data also indicate that the breakdown process is nonhomogeneous. Scale similarity is approximately valid, but, in the present data, a logarithmic correction is depicted. A probability distribution inside the family of infinitely divisible distributions is presented that compares well with experimental data.

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I. INTRODUCTION

The fundamental theory [1,2] about the local structure of turbulent fields is aimed to predict the scaling properties of turbulent velocity differences between two points separated by a given length scale, i.e., the cascade process, relating it to the rate of dissipation of the kinetic energy. Its major approximation stands on the assumption that the rate of dissipation is uniformly distributed inside the flow field.

The distribution of such a quantity is far from being uniform; it is actually strongly fluctuating, and intermittently distributed in the flow field. Then its average value is not sufficient for the description of the local structure of turbulence. This fact was pointed out by Landau [3] almost immediately after the formulation of the Kolmogorov-Obukhov 1941 theory and confirmed by experimental observations [4].

The first refined model of the internal structure of turbulence which takes into account the fluctuations of energy dissipation [5,6] introduces the concept of local average over spheres whose radius is equal to the length scale that similarity refers to. Even though the derived log-normal model was revealed as asymptotically incorrect [7], the concept of local average has then been used, and extended, for exploring the scaling of energy dissipation. Since then several models have been proposed, starting from experimental observations and phenomenological assumptions of the cascade dynamics; see [8,9] for reviews. Nevertheless, a rational and complete description for the scaling properties of the dissipation field is still to be developed and the phenomena leading to differences in the experimental observations must still be clarified [10,11].

The concept of scale similarity of random fields was developed in [7,12], where the cascade process was investigated by considering the ratio between the values of energy dissipation averaged over two spheres of different sizes contained one inside the other. These quantities, known as breakdown coefficients (or multipliers), appear as natural in understanding the scaling properties of random fields in several respects [13], and other representations, such as multifractal, can be derived from this [8,11].

The present work deals with the statistical description of breakdown coefficients with particular attention on their probability distribution. After an introduction to the problem, the dependence of the statistics from the local Reynolds number and the nonhomogeneity of the breakdown process is analyzed by experimental results (Sec. II). In Sec. III, the hypothesis of scale similarity is verified and a possible correction is presented. Then, in Sec. IV, the probability distribution of breakdown coefficients is considered inside the family of infinitely divisible distributions [14] and a new distribution is proposed.

II. BREAKDOWN COEFFICIENTS

Consider a non-negative random field $\varepsilon(x)$, which we will call the rate of energy dissipation, but it can be the square of any component of the velocity gradient tensor. Its experimental reading in time $\varepsilon(t)$ is assumed to be equivalent to a one-dimensional section of the field $\varepsilon(x)$ under the frozen-flow assumption $x = Ut$, with U equal to the mean velocity.

In the one-dimensional section of the field, let us consider two segments of sizes l and r, $l > r$, embedded one inside the other, and define the breakdown coefficients as

$$q_{r,l}(\Delta) = \frac{\varepsilon_r(x')}{\varepsilon_l(x)} \le \frac{l}{r} , \tag{1}$$

where

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$$\varepsilon_r(x) = \frac{1}{r}\int_{x-r/2}^{x+r/2} \varepsilon(s)ds$$

is the definition of local average. The inequality in (1) derives from the non-negativity of the field. The parameter Δ, defined by

$$\Delta = \frac{x'-x}{l-r}, \tag{2}$$

represents the relative displacement of the two segments, $\Delta = \pm\frac{1}{2}$ corresponds to the rightmost and leftmost inner segments, respectively, while $\Delta = 0$ means that the segments are centered at the same point. The statistical characteristics of breakdown coefficients (1) do not depend on the absolute position x when the field is homogeneous; this is correct for the small scales of a turbulent field assuming $r < l \ll L$, where L is a typical integral scale. At the same time, statistical characteristics, generally, depend on the local Reynolds number R_r, defined as $\langle \varepsilon \rangle^{1/3} r^{4/3} \nu^{-1}$ being ν the kinematic viscosity, which is equivalent to dependence on r/η, where $\eta = \nu^{3/4} \langle \varepsilon \rangle^{-1/4}$ is the Kolmogorov internal scale [15]; $\langle\ \rangle$ here and below means statistical averaging. The Δ dependence of the statistical characteristics cannot be dropped *a priori*, and only a symmetry for positive and negative values can be enforced for the local isotropy of the random field. This dependence defines the nonhomogeneity of the breakdown. A simple example of nonhomogeneity for a random sequence has been shown in [7]. It was also indicated in [7] that the correlation between ε_r and ε_l and their joint probability distribution depends on Δ. Under the natural assumption that the correlation function of the field $\varepsilon(x)$ decreases with the distance between points, it can be shown that the correlation between ε_r and ε_l is maximal for $\Delta = 0$ and minimal for $\Delta = \pm\frac{1}{2}$. Thus, the probability density function (PDF) of the breakdown coefficients (BDC) generally depends on Δ and we denote it by $W(q, l/r, \Delta)$ and its moments $\langle q_{r,l}^p(\Delta) \rangle = \int_0^{l/r} q^p W(q, l/r, \Delta) dq$.

In the following analyses we consider the experimental data set obtained by one of the authors (A.A.P.) in the return channel of the Central Aerodynamics Institute wind tunnel in Moscow [$175 \times 22 \times (20-32)$ m]. The detailed description, analysis, and verification of this data has been given elsewhere [16–18]. The Reynolds number based on the Taylor microscale is 3200 and the ratio between integral and viscous scales is $L/\eta \simeq 12\,000$. The data consist of approximately 4×10^6 measurements of both the longitudinal and transversal velocity derivatives. In what follows, the results obtained for the longitudinal derivative are reported, but no appreciable quantitative differences have been found in breakdown coefficients statistics for the two cases. We must also mention that independence from the sampling rate [10] has been verified in the presented results.

In the investigation of intermittency the structure of breakdown is explored inside the inertial range of turbulence. In this view it is assumed that the local scales are far enough from the dissipative scale $\eta \ll r < l \ll L$ such that the r/η dependence can be dropped. Nevertheless, few quantitative results are reported about how far scales must be from η. A limit of around $r/\eta \geq 500$ can be extrapolated from the results reported in [10].

In order to quantify the value of the local Reynolds number necessary to have a cascade process independent from the ratio r/r, we compute the probability density, and moments, varying r/η for several values of the breakdown ratio l/τ. In Fig. 1 the probability density of the breakdown coefficient is reported, at $\Delta = 0$, for $l/r = 2$ (top) and $l/r = 8$ (bottom), with the ratio r/η ranging from 50 to values for which l is comparable with the integral scale. It can be observed that for $l/r = 2$ convergence to a unique curve takes place for $r \simeq 1000\eta$, whereas for $l/r = 8$ the limit is reduced to about 300. In Fig. 2 the second- (top) and fourth- (bottom) order moments are plotted versus r/η for several values of l/r. Convergence for growing r/η, say, larger that 500, is observed (at least in first approximation). From this it must be emphasized that results obtained in intervals of size closer to η cannot be used directly for quantitative information about the inertial range breakdown. The relative-

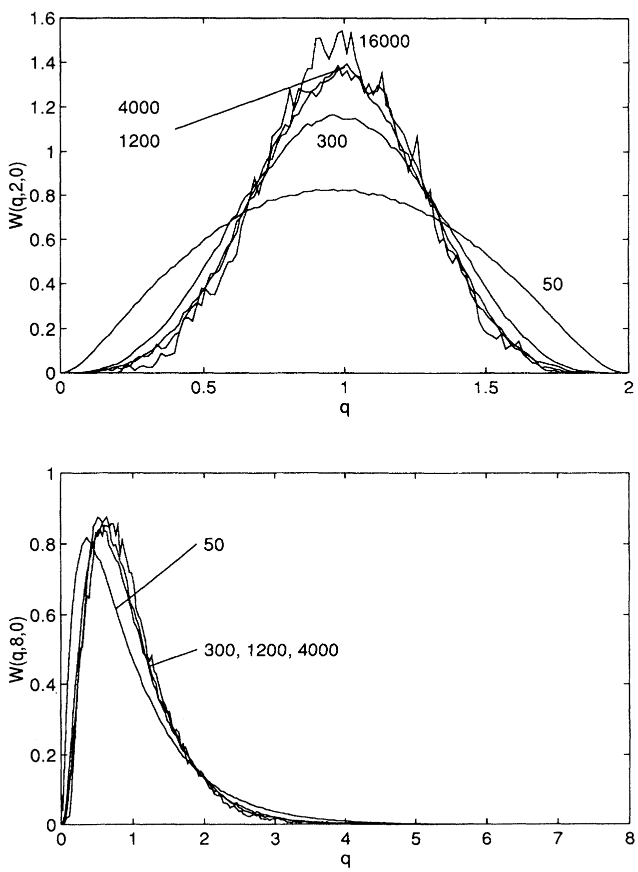

FIG. 1. Probability distribution function of centered breakdown coefficients ($\Delta = 0$) with $l/r = 2$ (top) and $l/r = 8$ (bottom). Various curves are for different values of r/η indicated in the pictures.

ly large limit value of the r/η ratio induces the necessity of data with very large Reynolds numbers, in general larger than what is necessary to identify inertial range behavior for velocities. An analogous dependence was observed in [10]. In what follows the results are presented for values of r/η large enough such as to neglect the dependence from it. Indicatively, it has been assumed that convergence to a unique statistic is obtained for $700\,\eta < r < l < 0.5L$.

The dependence of the breakdown process from the displacement parameter Δ has been often ignored in literature, leading to some controversy among apparently comparable results. Actually, it was shown in [7] that even a simple Markov random sequence can lead to this dependence, and no argument has ever been given in support of the breakdown homogeneity. The probability density functions for $l/r=2$ and $l/r=4$ are plotted in Fig. 3 (top) for $\Delta=0$ (continuous lines) and $\Delta=\frac{1}{2}$ (dashed lines). It can be clearly observed that the dependence from this parameter is significant. The continuous curves are comparable with the measurement reported in [10] obtained with $\Delta=0$; the dashed ones are more similar to measurements reported in [11,19]. The moments corresponding to the $l/r=2$ distributions are plotted in Fig. 3 (bottom). The distribution for $\Delta=\frac{1}{2}$ has a smaller maximum and more elongated tails (or larger high-order moments), indicating a more intermittent cascade process.

It is clear that the Δ dependence cannot be ignored and scaling considerations must be developed for a fixed value of Δ. In what follows the two limiting cases $\Delta=0$ and $\Delta=\frac{1}{2}$ will be considered.

III. VERIFICATION OF SCALE SIMILARITY

Inside the inertial range we can assume that the breakdown retains its universality while the corresponding Reynolds number is sufficiently large. Here the statistics of breakdown coefficients depends only on the scale ratio l/r and on the absolute value of the displacement $|\Delta|$.

In order to introduce the concept of scale similarity, let us consider an intermediate nested segment of size s, $r<s<l$, so that $q_{r,l}(\Delta)=q_{r,s}(\Delta)q_{s,l}(\Delta)$. Let us stress

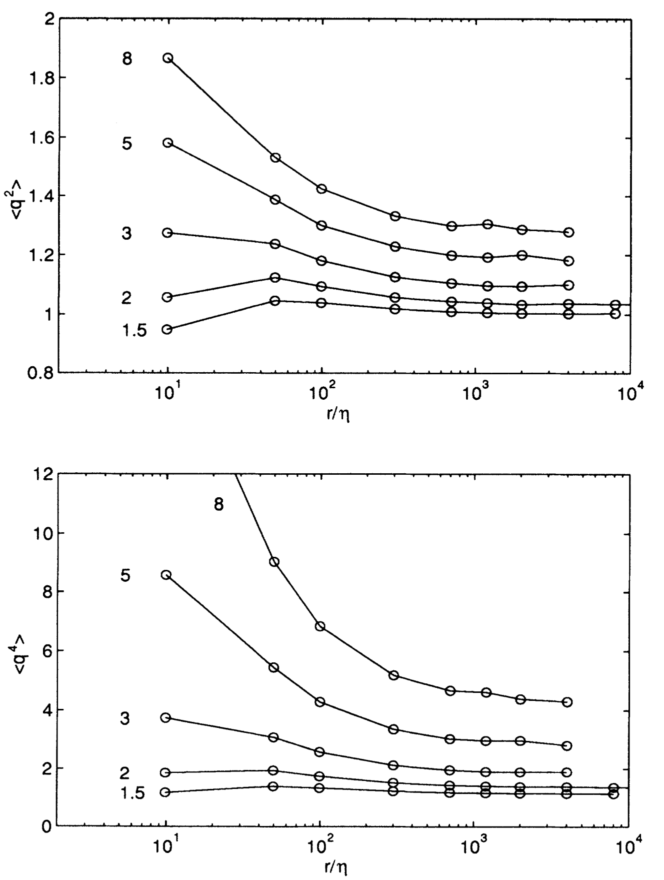

FIG. 2. Second- (top) and fourth- (bottom) order moments of breakdown coefficients as a function of r/η, for various ratios l/r, indicated in the pictures.

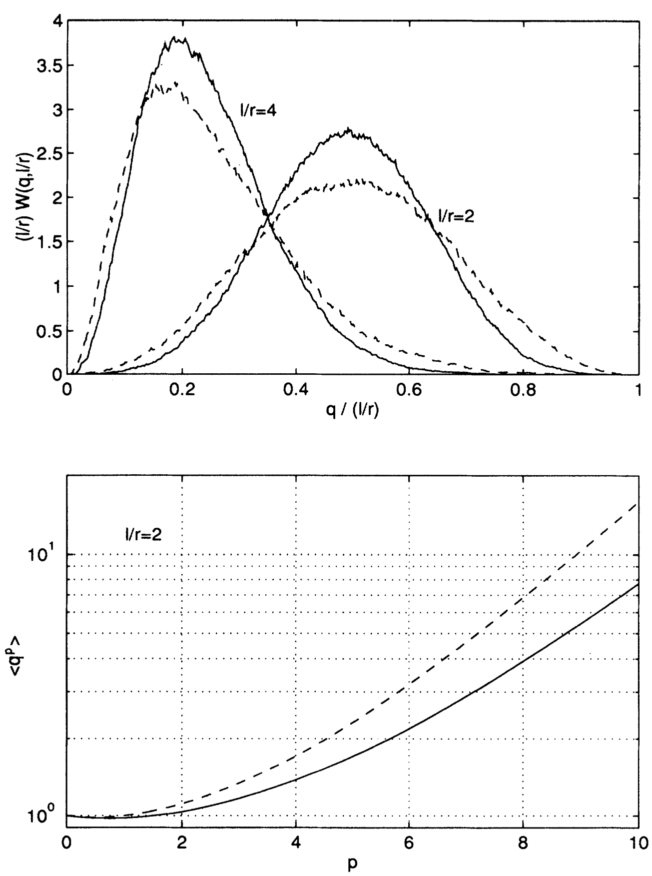

FIG. 3. Probability distribution function of breakdown coefficients with $l/r=2$ and 4 (top), and moments for $l/r=2$ (bottom), for displacement parameter $\Delta=0$ (continuous line) and $\Delta=0.5$ (dashed line).

that the relative displacement Δ should be kept the same through the breakdown process, otherwise we cannot expect similarity. Under the hypothesis

$$\langle q_{r,l}^{p}(\Delta)\rangle = \langle q_{r,s}^{p}(\Delta)\rangle \langle q_{s,l}^{p}(\Delta)\rangle \ , \quad (3)$$

it follows that moments must be expressed in accordance with scale similarity as

$$\langle q_{r,l}^{p}(\Delta)\rangle = \left[\frac{l}{r}\right]^{\mu(p,\Delta)} ; \quad (4)$$

with the conditions $\mu(0,\Delta)=0$, and [7,12]

$$\mu(p+\delta,\Delta)-\mu(p,\Delta)\le\delta \quad (\delta\ge 0) \ . \quad (5)$$

Notice that the $\mu(1)=0$ condition, used in Refs. [8,10,11,14,19–21], was derived [12] from independence of breakdown with respect to the parameter Δ and cannot be enforced [7].

As indicated in [8], condition (3) is sufficient and necessary for scale similarity (4). Condition (3) follows from the stronger condition of independence of successive breakdown coefficients $q_{r,s}$ and $q_{s,l}$. Before moving to verification of Eq. (4) we start by verifying the hypothesis leading to it.

Independence of successive breakdown coefficients ensures the fulfillment of scale similarity. In order to verify

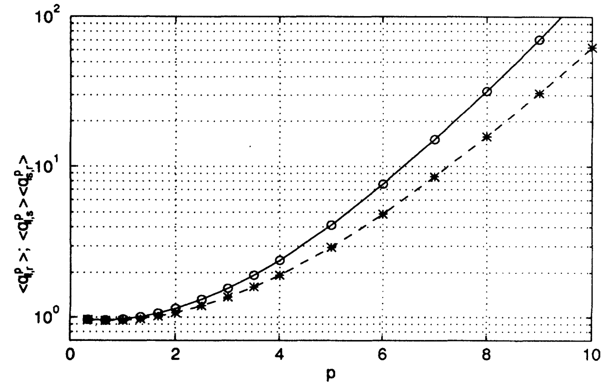

FIG. 5. Moments of breakdown coefficients $\langle q_{r,l}^{p}\rangle$ (continuous line with circles) and product $\langle q_{r,s}^{p}\rangle\langle q_{s,l}^{p}\rangle$ (dashed line with stars) for $l/s=s/r=2$.

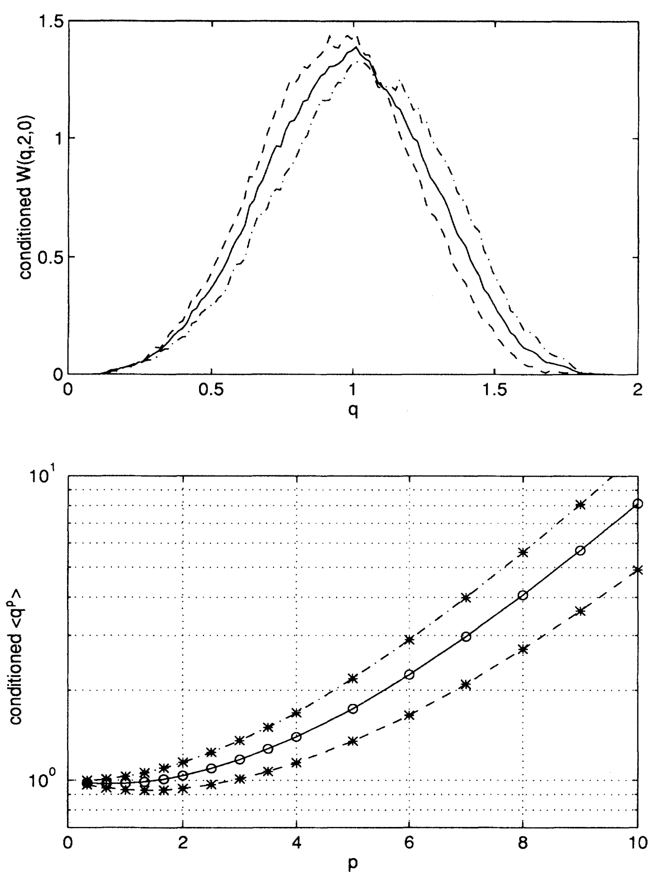

FIG. 4. Conditional probability distribution function (top) and moments (bottom) of breakdown coefficients $q_{r,s}$, conditioned to the value of $q_{s,l}$ for $l/s=s/r=2$ (BDC are computed by ratios of averaged energy dissipation measured simultaneously in three intervals contained one in another and centered, $\Delta=0$). Unconditional (continuous line), conditioned to $q_{s,l}\le 1$ (dashed line), conditioned to $q_{s,l}>1$ (dash-dotted line).

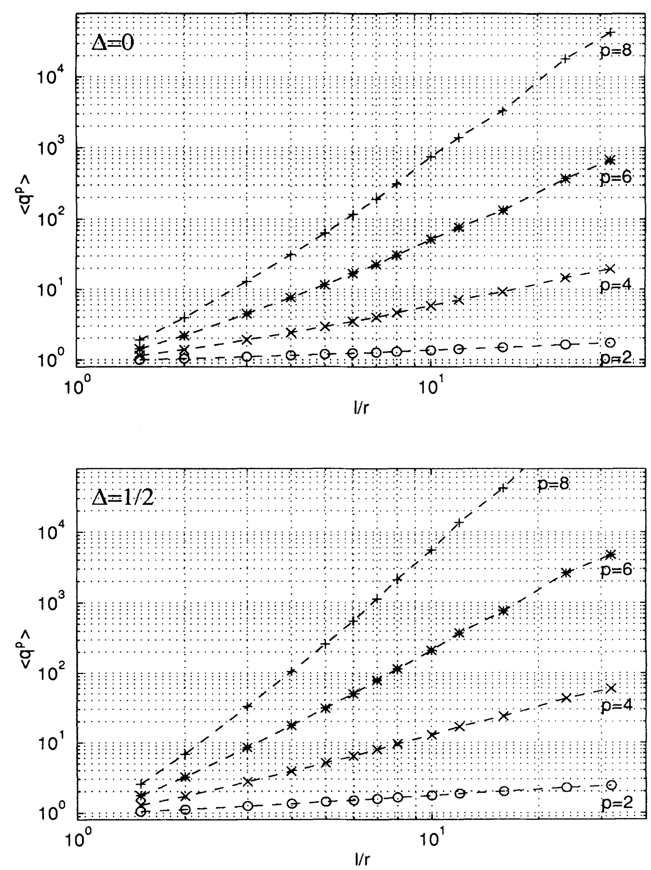

FIG. 6. Moments of breakdown coefficients $\langle q_{r,l}^{p}\rangle$, $p=2,4,6,8$, as a function of l/r. The dashed lines connect the experimental data.

this we consider the statistics of $q_{r,s}$, in the arbitrarily chosen case $s/r=l/s=2$, conditioned with the value of $q_{s,l}$ larger and smaller than 1. In Fig. 4 (top) we report the probability distribution of $W(q,s/r=2,0)$ (continuous curve) and the same function conditioned by $q_{s,l}\le 1$ (dashed) and $q_{s,l}>1$ (dash-dotted). Corresponding moments are reported below. We can clearly observe systematical differences between the two statistics, indicating a dependence of successive coefficients. This fact has already been noticed in [19]. It can be seen that for $q_{s,l}<1$ the probability distribution of $q_{r,s}$ is shifted towards smaller values; for $q_{s,l}>1$ the shift is reversed. Let us stress that condition (3) is slightly weaker than complete independence and is still sufficient for scale similarity (see also [19]). In Fig. 5 we report the two sides of Eq. (3) for $s/r=l/s=2$, $\Delta=0$; here we can see that hypothesis (3) is not supported by the used data. However, it is not clear how the discrepancies in Eq. (3), as measurable from Fig. 5, influence the verification of scale similarity (see also [19]).

So, even without an *a priori* guarantee, we can check the validity of Eq. (4) directly. In Fig. 6 we report moments of breakdown coefficients versus scale ratio l/r for moment order $p=2,4,6,8$, and $\Delta=0$ (top), $\Delta=\frac{1}{2}$ (bottom). It must be pointed out that, for scale ratio larger than about 8, there is no way of keeping both segments l and r inside the suggested limits $700\eta<r<l<0.5L\simeq 6000\eta$, expressed in the preceding section; data reported for larger values have been included to extend the l/r domain and for verification of trends but must be considered with caution. From the pictures we can see that scale similarity, expressed by Eq. (4), approximates closely the actual physics. Nevertheless, the agreement is not perfect and a small curvature can be seen in the reported graphs, in particular keeping in mind that curves must converge to the point $(l/r=1,\langle q^p\rangle=1)$. This behavior has been verified to occur independently from the local Reynolds number, when it is sufficiently large, as explained in Sec. II. We must mention that a better agreement with self-similar scaling could be noticed in a limited range of r/η, about 10–100, but with a substantial dependence of the exponents from the value of r/η itself.

The rest of the present work is focused on describing the probability distribution of breakdown coefficients. In

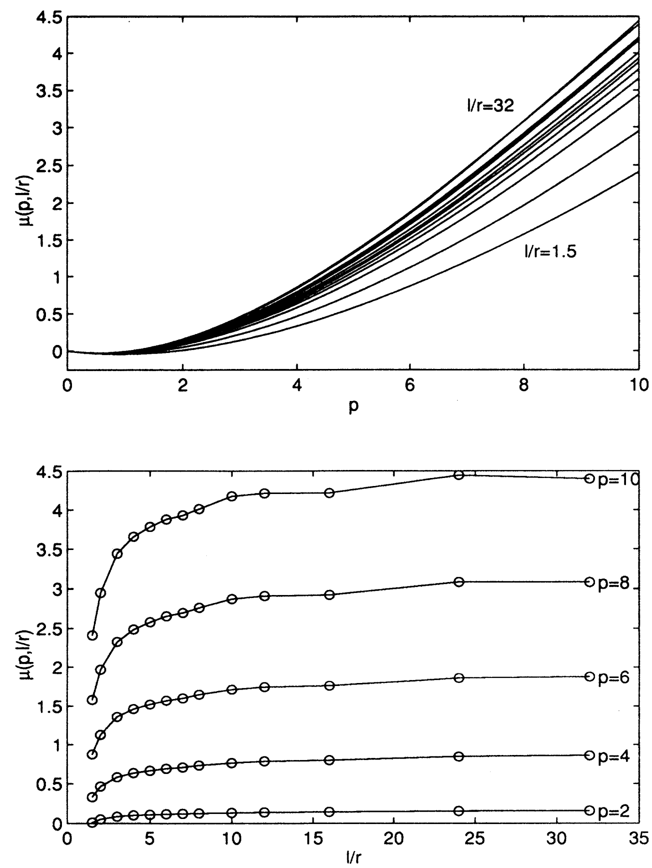

FIG. 7. Scaling exponent $\mu(p,l/r,0)$ as a function of moment order p for various scale ratios (top), as a function of scale ratio for various moment order (bottom).

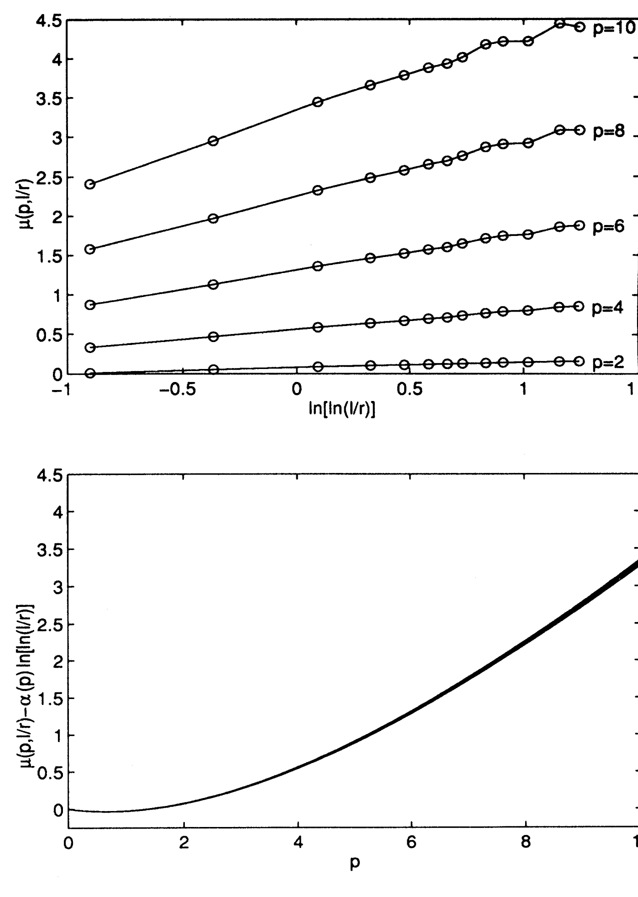

FIG. 8. Scaling exponent $\mu(p,l/r,0)$ as a function of the double logarithm of scale ratio for various moment order (top); and its value, subtracted by the logarithmic variation with scale ratio, as a function of moment order for various scale ratios (bottom).

this view we wish to look at moments (4) in an alternative perspective: for each value of the ratio l/r we can write

$$\mu\left[p,\frac{l}{r},\Delta\right]=\log_{l/r}\left\{\int_0^{l/r} q^p W\left[q,\frac{l}{r},\Delta\right]dq\right\}; \qquad (6)$$

where the l/r dependence has been included on the left side, and it would disappear in the case of scale similarity. The results from Eq. (6) are reported in Fig. 7, for $\Delta=0$. It can be noticed that even if scale similarity represents a first approximation to the actual scaling, a relevant additional l/r dependence is present. This dependence is plotted in Fig. 7 (bottom), for moments. After an extensive analysis in which several possible additional l/r dependencies have been tested, the unique functional agreement has been found with a bilogarithmic dependence of the exponent μ, as shown by the alignment of the experimental points in Fig. 8 (top). After this, moments of breakdown coefficients can still be expressed by Eq. (4), at least for $1.5\le 1/r\le 12$, with

$$\mu\left[p,\frac{l}{r},\Delta\right]=\mu_0(p,\Delta)+\alpha(p,\Delta)\ln\left[\ln\left[\frac{l}{r}\right]\right]; \qquad (7)$$

The two unknown functions $\mu_0(p,\Delta)$ and $\alpha(p,\Delta)$ can be derived by fitting the experimental data, where $\mu_0(p,\Delta)$ corresponds to $\mu(p,l/r,\Delta)$ with $l/r=e$ (e being the base of natural logarithm). In the present case the functions $\mu_0(p,\Delta)$ and $\alpha(p,\Delta)$ have been obtained by a best fit of the data reported in Fig. 8 (top), for $l/r\le 12$. The collapse into approximately one curve is shown in Fig. 8 (bottom), where the exponent $\mu(p,l/r,\Delta)$, subtracted by the additional term, is plotted. For future reference, the present estimates for the functions $\mu_0(p,\Delta)$ and $\alpha(p,\Delta)$ are reported in Fig. 9 for $\Delta=0$ (continuous line) and $\Delta=\frac{1}{2}$ (dashed line).

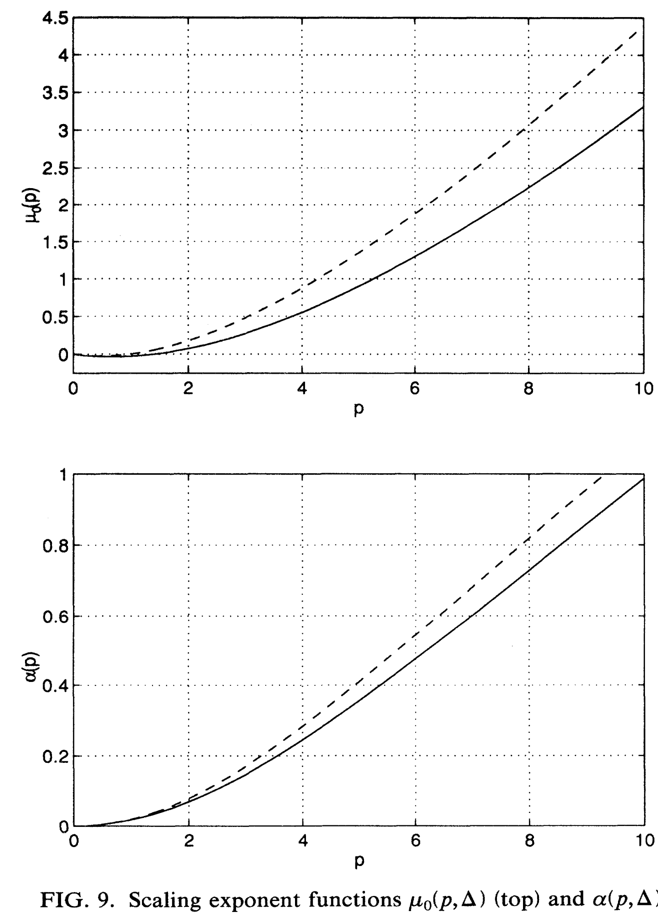

FIG. 9. Scaling exponent functions $\mu_0(p,\Delta)$ (top) and $\alpha(p,\Delta)$ (bottom), obtained by best fit of experimental data. Displacement parameter $\Delta=0$ (continuous line), $\Delta=\frac{1}{2}$ (dashed line).

Formula (7) corresponds to a multiplicative correction to scale similarity (4) given by $\ln(l/r)^{\alpha(p,\Delta)\ln(l/r)}$. This logarithmic correction cannot be assumed as a general character of breakdown until confirmed by further results. It could be connected to limitation of Reynolds number in the present data and we hope it will stimulate measurements, in particular at larger R_e, to verify scale similarity and, possibly, logarithmic correction.

In constructing a model, it is useful to have in mind restrictions, which follows from the definition (1) of breakdown coefficients. Limiting ourselves to the case $\Delta=0$ and $\Delta=\pm\frac{1}{2}$, we have the following identities:

$$q_{r,l}\left[\frac{1}{2}\right]=\frac{l}{r}-\left[\frac{l}{r}-1\right]q_{l-r,l}\left[-\frac{1}{2}\right], \qquad (8)$$

$$q_{r,l}(0)=\frac{l}{r}-\frac{1}{2}\left[\frac{l}{r}-1\right]\left[q_{(l-r)/2,l}\left[-\frac{1}{2}\right]+q_{(l-r)/2,l}\left[+\frac{1}{2}\right]\right]. \qquad (9)$$

Statistical constraints are obtained by taking the p power of both sides in (8) and (9), statistical averaging, and taking into account local isotropy (i.e., dependence of statistics from the sign of Δ). This procedure gives relations among various moments at different (conjugate) scale ratios. In the simplest case $p=1,3$, $l/r=2$, we obtain from (8) and (9)

$$\mu(1,2,\tfrac{1}{2})=0,\quad 2^{\mu(1,2,0)}=2-4^{\mu(1,4,1/2)},$$
$$2^{\mu(3,2,1/2)}=3\times 2^{\mu(2,2,1/2)}-2. \qquad (10)$$

For $p=2$ identity (8) was used [14] in order to show that with $\Delta=\pm\frac{1}{2}$ we do not have scale similarity. For simplicity it was assumed $\mu(1,l/r,\pm\frac{1}{2})=0$. Without this assumption, denoting $l/r=\lambda$, we have

$$\lambda^{\mu(2,\lambda,1/2)}=\lambda^2-2\lambda(\lambda-1)\left[\frac{\lambda}{\lambda-1}\right]^{\mu(1,\lambda/(\lambda-1),1/2)}+(\lambda-1)^2\left[\frac{\lambda}{\lambda-1}\right]^{\mu(2,\lambda/(\lambda-1),1/2)}. \qquad (11)$$

We see that if μ does not depend on scale ratio, Eq. (11) cannot be satisfied for arbitrary λ (apart from the trivial solution corresponding to a uniformly distributed rate of dissipation field). The exception is $\lambda=2$. Thus, generally speaking, there is no scale similarity for $\Delta=\pm\frac{1}{2}$. The best candidate for scale similarity is, probably, breakdown with $\Delta=0$ for very high Reynolds number.

Model (7) with μ_0 and α, obtained by fit of the data, was tested with the use of Eq. (8) in the range $1<l/r\le 8$ for $p=1,2,3$. A relative error was found from 0.5–5% which is related to the empirical estimates of the functions appearing in (7). An analytical modeling for the dependence of moments from p, Δ, and possibly l/r, can be verified and improved in future for broader data sets by an explicit consideration of such constraints.

IV. PROBABILITY DISTRIBUTION OF BREAKDOWN COEFFICIENTS

The present section is devoted to the description of the probability distribution $W(q,l/r,\Delta)$ of the breakdown coefficients aiming to verify the correctness of modeling and to identify possible directions to develop optimal models.

The concept of scale similarity has been embedded into the theory of infinitely divisible probability distribution in [14]. When breakdown is scale similar, it has been shown that if the variable $z_{r,l}=-\ln(r/lq_{r,l})$ has infinitely divisible distribution, concentrated on the interval $[0,\infty)$, then all properties for the function $\mu(p,\Delta)$ [see, in particular, Eq. (5)] and for the probability $W(q,l/r,\Delta)$ are automatically satisfied. In this case an alternative approach to modeling is given by choosing a measure F on the interval $(0,\infty)$ such that $(1+x)^{-1}$ is integrable with respect to it. Given such a measure, the scaling exponent is [14]

$$\mu(p)=\kappa p-\int_0^{\infty}\frac{1-e^{-px}}{x}F(dx)\;; \tag{12}$$

and the probability distribution is then given by

$$W\left[q,\frac{l}{r}\right]=\frac{1}{2\pi q}\int_{-\infty}^{+\infty}\exp\left\{-is\ln(q)+\mu(is)\ln\left[\frac{l}{r}\right]\right\}ds\;. \tag{13}$$

The problem of choosing a model is then reduced, in the case of scale similarity, to the determination of the measure F. In particular, a measure with density

$$F'(x)=Ae^{-x/\sigma},\quad A>0,\ \sigma>0 \tag{14}$$

corresponds to a log-gamma distribution for breakdown coefficients, proposed in [20]. An unusual extension of (14) will be used below in the form

$$F'(x)=A(e^{-x/\sigma_1}-e^{-x/\sigma_2}),\quad \sigma_1>\sigma_2\;. \tag{15}$$

Unlike the traditional superposition of measures with different parameters, Eq. (15) corresponds to a *difference* between two measures and is a measure only with additional condition on parameters. The behavior of this measure near $x\sim 0$ is quite different from (14). The model (15) can be called diexponential, it has three parameters and, in this sense, is more flexible than model (14). Let us note that the δ measure $F'(x)=C\delta(x-a)$ with positive constants C and a, corresponds to models presented in [21] and [22].

The free parameter κ in Eq. (12) is related to the asymptotic behavior of $\mu(p)$ for large p. Following [14], we assume that the probability distribution $W(q,l/r,\Delta)$ has no gap in the interval $[0,l/r]$. In fact, our data support this assumption (see also below). From the definition (6) we write:

$$\mu\left[p,\frac{l}{r},\Delta\right]\le\log_{l/r}\left\{\left[\frac{l}{r}\right]^p\int_0^{l/r}W\left[q,\frac{l}{r},\Delta\right]dq\right\}=\log_{l/r}\left\{\left[\frac{l}{r}\right]^p\right\}=p\;, \tag{16}$$

where normalization of W is used. In the case of scale similarity, this inequality follows also from condition (5). Defining

$$h=\lim_{p\to\infty}\frac{1}{p}\mu(p,l/r,\Delta)\;,$$

we have also

$$\begin{aligned}h&\ge\lim_{p\to\infty}\frac{1}{p}\log_{l/r}\left\{\int_{\lambda_0}^{l/r}q^pW\left[q,\frac{l}{r},\Delta\right]dq\right\}\\&\ge\lim_{p\to\infty}\frac{1}{p}\log_{l/r}\left\{\lambda_0^p\int_{\lambda_0}^{l/r}W\left[q,\frac{l}{r},\Delta\right]dq\right\}\\&=\log_{l/r}(\lambda_0)+\lim_{p\to\infty}\frac{1}{p}\log_{l/r}\left\{\int_{\lambda_0}^{l/r}W\left[q,\frac{l}{r},\Delta\right]dq\right\}\\&=\log_{l/r}(\lambda_0)\end{aligned} \tag{17}$$

TABLE I. Parameter values of the probability density functions discussed in the text.

		Exponential		Diexponential		
l/r	Δ	A	σ	A	σ_1	σ_2
2	0	8.47	0.129	8.80	0.126	0.161×10^{-2}
2	$\frac{1}{2}$	5.29	0.208	5.90	0.192	0.613×10^{-2}
4	0	6.14	0.182	7.65	0.156	0.113×10^{-1}
4	$\frac{1}{2}$	3.71	0.314	5.89	0.221	0.319×10^{-1}
8	0	5.07	0.226	8.44	0.159	0.279×10^{-1}
8	$\frac{1}{2}$	2.97	0.413	13.3	0.177	0.932×10^{-1}

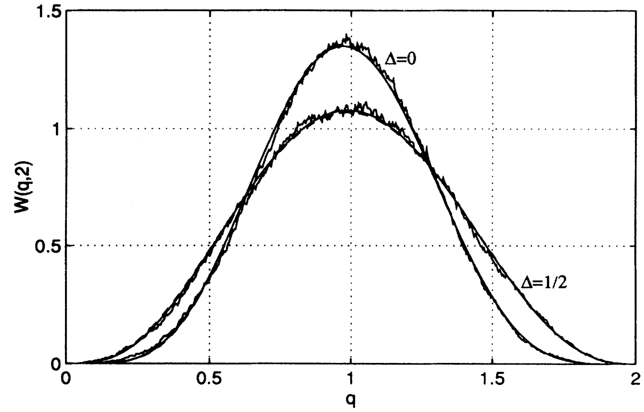

FIG. 10. Probability density function, $l/r=2$ and $\Delta=0,\frac{1}{2}$: experimental data (continuous rough line), diexponential model (continuous line), exponential model (dashed line).

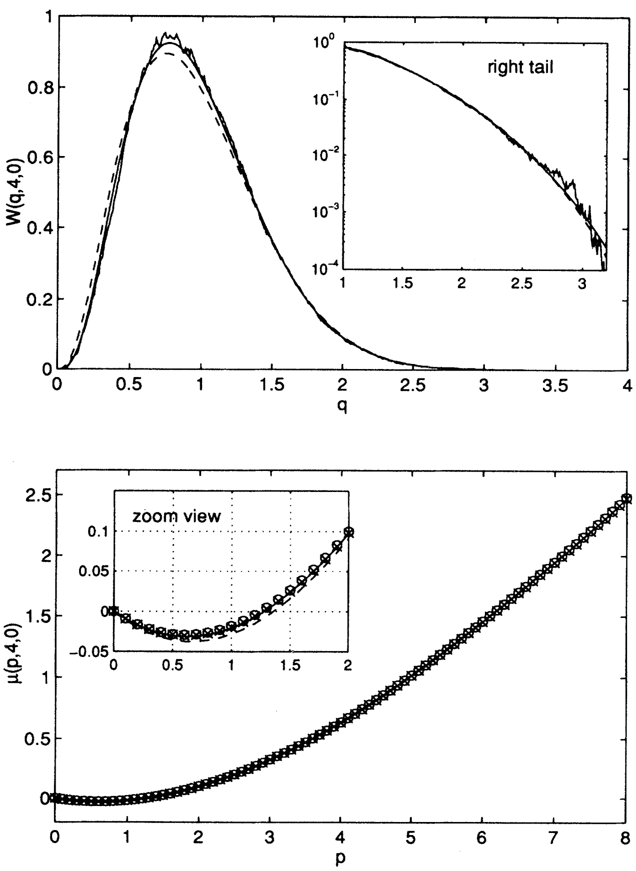

FIG. 11. Probability density function, $l/r=4$ and $\Delta=0$: experimental data (continuous rough line), diexponential model (continuous line), exponential model (dashed line). Note that the right tail is presented log-linearly, which changes curvature. Corresponding scaling exponent (bottom): experimental data (circles), its approximation by logarithmically corrected model (crosses), diexponential model (continuous line), exponential model (dashed line).

for any $\lambda_0 < l/r$. Here we assumed that there is no probability gap. From (16) and (17) we obtain the no-gap condition [14] $h=1$. This result also has the implication for the asymptotic scaling exponents ζ_n of velocity increments in the inertial range. The usual connection between exponents for velocity increments and dissipation, $\zeta_n = n/3 - \mu(n/3)$, given by the refined similarity hypothesis of Kolmogorov, leads to the condition $\zeta_n/n \to 0$ with $n \to \infty$. The model presented in Ref. [21] has $h=2/3$ and implies $\zeta_n/n \to 1/g$ with $n\to\infty$.

Using this asymptotic condition for the models (14) and (15) we get in (12) the constant $\kappa=1$. The scaling exponents corresponding to Eqs. (14) and (15) can be obtained analytically as

$$\mu(p) = p - A\ \ln(p\sigma + 1) \tag{18}$$

and

$$\mu(p) = p - A\ \ln\left[\frac{p\sigma_1+1}{p\sigma_2+1}\right], \tag{19}$$

respectively.

The present data do not exactly satisfy scale similarity. Nevertheless, we assume that infinitely divisible models are still appropriate, at least as a good starting point, tak-

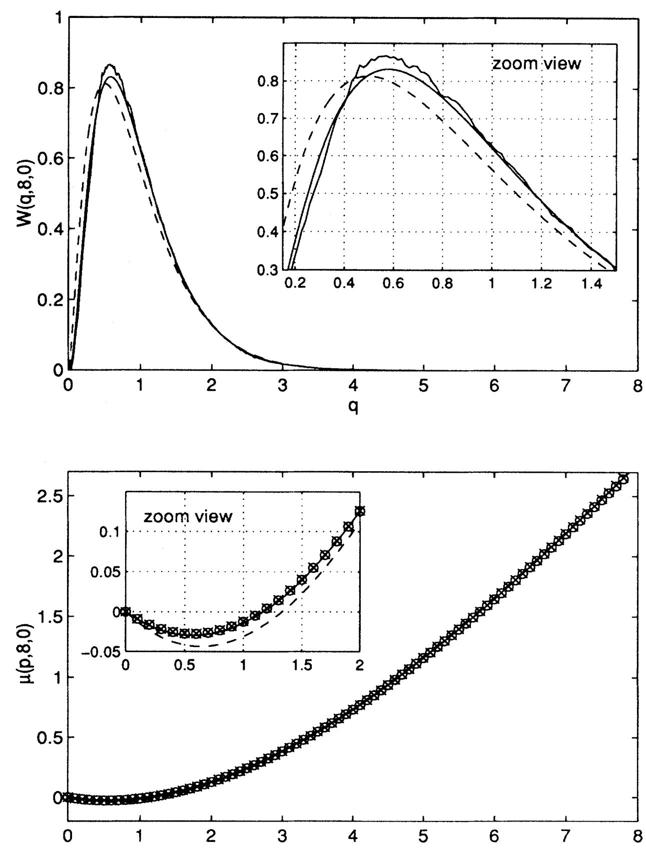

FIG. 12. The same as Fig. 11, but with $l/r=8$.

ing into account the logarithmic l/r dependence observed in the preceding section. In order to test the accuracy of models (14) and (15), we consider the scaling exponent $\mu(p,l/r,\Delta)$ given by formula (7) with the functions $\mu_0(p)$ and $\alpha(p)$ plotted in Fig. 9. By this, parameters in formula (18) and (19) can be evaluated by the least-square error minimization procedure, without any further use of experimental data after the estimate of $\mu_0(p)$ and $\alpha(p)$. The parameters of the probability distribution used in the following results, are reported in Table I.

The probability distribution obtained for $l/r=2$ is plotted in Fig. 10 for $\Delta=0$ and $\frac{1}{2}$. The data (rougher curve) and the result from the model (15), (19) are plotted with continuous line, model (14), (18) is reported with the dashed line. It can be noticed that the models are practically coincident (actually $\sigma_2\simeq 0$) and adapt well on the experimental data. The curve for $\Delta=\frac{1}{2}$ appears identical to that of Ref. [11].

The probability distribution for $l/r=4$, $W(q,4,0)$ is reported in Fig. 11 (top), with the same lines as in Fig. 10. The corresponding scaling exponent is plotted below: the experimental points are represented by a circle, the outcome from formula (7) with crosses, the diexponential model (15) corresponds to the continuous line, the exponential model (14) is represented by the dashed line. In this case the diexponential model adapts well on the experimental data whereas the simpler exponential model is less accurate. It must be noticed that both models well capture the tail of the distribution, both being consistent with the asymptotic consequence (16), (17). The outcome for $l/r=8$, $\Delta=0$, is described by Fig. 12. The probability distribution (top) is still well represented by the diexponential model. The scaling exponent $\mu(p,8,0)$ is reported below. Looking at the last two pictures (in particular, at the zoom views) we can conclude that extremely small errors in μ lead to significant deviation in the probability distribution. Moreover, very similar curves $\mu(p)$ can correspond to very different probability distributions, as is the case of the binary model [23], where it is shown that a good fit of experimental data on $\mu(p)$ can be obtained with a model corresponding to an unrealistic probability distribution given by two δ functions.

Model (15) improves significantly the description of the data on the probability distributions at values of the scale ratio larger than 2, in comparison with the simpler model (14). This fact may suggest that more sophisticated measures are necessary to describe the cascade process at larger scale ratios. However, the presumable increasing independence of successive breakdown coefficients at increasing values of the scale ratio could lead to a simple description of the asymptotic behavior. The representativeness of the diexponential model for intermittency should be tested with different experimental data.

V. CONCLUSIONS

The description of intermittency in terms of a breakdown process has been considered. It has been shown that statistics of breakdown coefficients converge to a unique inertial representation when the local Reynolds number is sufficiently large, its value being substantially higher than what is usually compatible with velocity statistics in the inertial range. For this reason, results obtained at a smaller distance from the dissipative scale can describe the breakdown process but cannot be assumed as quantitative for an inertial range cascade. It has also been shown that the breakdown process is nonhomogeneous, being dependent on the relative displacement (Δ). Different data sets can be compared only at corresponding values of the displacement parameter, and simplification of the theory, connected with breakdown homogeneity, must be revised.

The hypothesis of scale similarity holds only approximately. A logarithmic correction has been depicted. It may be related to limitation in the Reynolds number in the processed data. We hope that this feature will stimulate its verification in existing sets of data and measurements at larger Reynolds number. Verification of scale similarity and the possibly of indicated logarithmic dependence plays a central role in the future development of the theory.

Models for intermittency have been chosen inside the family of infinitely divisible probability distributions. A measure, based on the difference between two exponential measures, has been shown to well approximate the present data. Comparisons have been performed on the scaling exponent and on the probability distribution. The latter is more adequate in verification of models whereas extremely small variations in the scaling exponent may correspond to substantial differences in the probability distribution.

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